Algebra Qualifying Exam II (May 2019)

1. (10 points) Let $R = k[x, y]$ be a polynomial ring in two variables over a field k. Find a module over R, which is finitely generated torsion free but not free.

2. (10 points) Let a be a rational number such that $-\frac{1}{2} < a < \frac{1}{2}$. Prove that $\tan(a\pi)$ is algebraic over $\mathbb{Q}$.

3. (10 points) Let $\zeta_7 = e^{2\pi\sqrt{-1}/7}$ and let $E = \mathbb{Q}[\zeta_7]$. Explicitly determine all the fields L such that $\mathbb{Q} \subset L \subset E$.

4. (10 points) Consider the polynomial

$$f(x) = 7x^5 + 4x^2 + 6x + 30.$$

   Let $\alpha \in \mathbb{C}$ be a complex number such that $f(\alpha) = 0$.

   (a) Prove that the polynomial $f(x)$ is irreducible in $\mathbb{Q}[x]$.

   (b) Let $\beta = \sqrt[16]{2}$. Prove that $\alpha \notin \mathbb{Q}[\beta]$.

5. (10 points) Let K be a finite field, and let $f \in K[x_1, \ldots, x_n]$ be a polynomial in n variables. Let E be a finite field extension of K. Define the sum

$$S_f = \sum_{(a_1,\ldots,a_n)\in E^n} f(a_1, \ldots, a_n).$$

   Prove that $S_f \in K$.

6. (10 points) Let E be the splitting field of the polynomial $f(x) = x^4 - 2x^2 - 2$ over $\mathbb{Q}$. Prove that the Galois group $\mathrm{Gal}(E/\mathbb{Q})$ is isomorphic to the dihedral group

$$D_4 := \left\langle r, s \;\middle|\; r^4 = 1,\ s^2 = 1,\ srs = r^{-1} \right\rangle$$